ANNALS OF MATHEMATICS.

VOL. III. DECEMBER, 1887. No. 6.

ON THE ORBIT OF HYPERION.

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1. The principal difficulty in the integration of the equations of motion in the case of the problem of three bodies arises in the integration of terms involving the inverse powers of the distance between the disturbed and disturbing bodies. When the ratio between the radius vectors is not too near unity, the inverse powers referred to can be developed in rapidly converging series, in terms of multiples of the elongation. When, however, these ratios do not differ greatly from unity, the convergence of the series mentioned is very slow. If, in addition, the mean motions of the two bodies are nearly commensurate, the ordinary methods of solving the problem become inapplicable.

Such a case presents itself in the determination of the perturbations of Hyperion produced by Titan. On the other hand, the mutual inclination of the orbits of these two satellites is so small as to have eluded detection; the eccentricity of the orbit of Titan is less than 0.03; and the position of the apo-saturnium of Hyperion so nearly coincides (at least at present) with the point of conjunction of the two satellites, as to give rise to a suspicion that the eccentricity of its orbit is small, and that the apparent eccentricity is principally due to the perturbations produced by Titan.

2. *First Approximation.*—These circumstances have suggested* the propriety of first investigating that part of the disturbance which may be determined by neglecting the mutual inclination and the eccentricities of the orbits of both bodies. On this hypothesis Tisserand has found that, assuming the mass of Titan $m' = \frac{1}{10750}$, the motion of Hyperion may be represented, to terms of the first order with respect to the mass, by the formulæ,

$$\frac{r}{a} = 1 - 0.0004 \cos \quad (l' - l) - 0.0014 \cos 2\,(l' - l)$$
$$+ 0.1000 \cos 3\,(l' - l) + 0.0006 \cos 4\,(l' - l),$$

*Tisserand. Sur un Cas Remarquable du Problème des Perturbations. Comptes-Rendus, 1886, 2e Semestre, p. 446.

Hill. Coplanar Motion of Two Planets, One Having a Zero Mass. ANNALS OF MATHEMATICS. Vol. III. p. 65.

$$\begin{aligned} w = l + & \ 10' \sin \ (l' - l) + 13' \sin 2 (l' - l) \\ & - 683' \sin 3 (l' - l) - \ 3' \sin 4 (l' - l), \end{aligned} \tag{1}$$

where r is the radius vector of its orbit,

a	mean distance from Saturn,
w	true longitude in orbit,
l	mean longitude in orbit,
l'	mean longitude in orbit of Titan.

3. *Second Approximation.*—As a second approximation let us assume

$$r = a(1 + a_1 \cos \theta + a_2 \cos 2 \theta + \ldots) = a(1 + \sigma), \tag{2}$$

$$\frac{dw}{dt} = n(1 + n_1 \cos \theta + n_2 \cos 2 \theta + \ldots) = n(1 + \tau), \tag{3}$$

where r and w are the radius vector and longitude in orbit of Hyperion, $\theta = l' - l = (n' - n) t$ is the mean angular distance between the radius vectors of Titan and Hyperion, a and n are the mean values of r and w, and a_1, n_1, etc. are constants to be determined.

The differential equations of motion may be written

$$r^2 \frac{dw}{dt} = m'k^2 \int S r dt + C, \tag{4}$$

$$\frac{d^2 r}{dt^2} - r \left(\frac{dw}{dt} \right)^2 + \frac{k^2}{r^2} = m'k^2 R, \tag{5}$$

where k is the constant of the system as derived from the orbit of Titan, m' the mass of Titan, $m'k^2R$ and $m'k^2S$ the components in the plane of the orbit of the disturbing force in the direction of and perpendicular to the disturbed radius vector, and C is a constant of integration. As is well known, when the orbit is nearly circular, C is approximately equal to $k\sqrt{a}$; say $k\sqrt{[a(1 - \nu)]}$, where ν is a small quantity.

4. Substituting in (4) the values of r and dw/dt given by (2) and (3), and dividing by $a^2 n$, we have

$$\frac{r^2}{a^2 n} \frac{dw}{dt} = 1 + 2\sigma + \tau + \sigma^2 + 2\sigma\tau + \sigma^2\tau = \frac{k\sqrt{(1 - \nu)}}{a^{\frac{3}{2}} n} + m' \frac{k^2}{a^2 n} \int S r dt, \tag{6}$$

in which, if $\Sigma = \sum_{i=1}^{i=\infty}$, i being integral and positive,

$$\sigma = a_1 \cos \theta + a_2 \cos 2\theta + \ldots = \Sigma a_i \cos i\theta,$$

$$\tau = n_1 \cos \theta + n_2 \cos 2\theta + \ldots = \Sigma n_i \cos i\theta,$$

$$\sigma^2 = \tfrac{1}{2}\Sigma a_i^2 + \Sigma a_i a_{i+1}\cos\theta + (\tfrac{1}{2}a_1^2 + \Sigma a_i a_{i+2})\cos 2\theta$$
$$+ (a_1 a_2 + \Sigma a_i a_{i+3})\cos 3\theta + (\tfrac{1}{2}a_2^2 + a_1 a_3 + \Sigma a_i a_{i+4})\cos 4\theta$$
$$+ (a_1 a_4 + a_2 a_3 + \Sigma a_i a_{i+5})\cos 5\theta + \ldots .$$

5. Comparing (2) and (3) with Tisserand's values of r/a and w, we are led to infer that it will be proper, for the present, to consider a_3 and n_3 as quantities of the first order; a_1, a_2, a_4, n_1, n_2, n_4, as quantities of the third order; a_5, a_6, a_7, a_8, n_5, n_6, n_7, n_8 as quantities of the fourth order; and neglect the remaining coefficients of σ and τ.

We have, then, to terms of the fourth order,

$$\sigma = a_1\cos\theta + a_2\cos 2\theta + \ldots + a_8\cos 8\theta, \tag{7}$$
$$\tau = n_1\cos\theta + n_2\cos 2\theta + \ldots + n_8\cos 8\theta,$$
$$\sigma^2 = \tfrac{1}{2}a_3^2 + a_3(a_2 + a_4)\cos\theta + a_1 a_3\cos 2\theta + a_1 a_3\cos 4\theta \tag{8}$$
$$+ a_2 a_3\cos 5\theta + \tfrac{1}{2}a_3^2\cos 6\theta + a_3 a_4\cos 7\theta,$$
$$2\sigma\tau = a_3 n_3 + [(a_2 + a_4)n_3 + a_3(n_2 + n_4)]\cos\theta$$
$$+ (a_1 n_3 + a_3 n_1)\cos 2\theta + (a_1 n_3 + a_3 n_1)\cos 4\theta$$
$$+ (a_2 n_3 + a_3 n_2)\cos 5\theta + a_3 n_3\cos 6\theta$$
$$+ (a_3 n_4 + a_4 n_3)\cos 7\theta,$$
$$\sigma^2\tau = \tfrac{3}{4}a_3^2 n_3\cos 3\theta + \tfrac{1}{4}a_3^2 n_3\cos 9\theta.$$

Whence (6) becomes

$$\frac{r^2}{a^2 n}\frac{dw}{dt} = 1 + \tfrac{1}{2}a_3^2 + a_3 n_3$$
$$+ [2a_1 + n_1 + a_3(a_2 + a_4 + n_2 + n_4) + (a_2 + a_4)n_3]\cos\theta$$
$$+ [2a_2 + n_2 + a_3(a_1 + n_1) + a_1 n_3]\cos 2\theta$$
$$+ [2a_3 + n_3 + \tfrac{3}{4}a_3^2 n_3]\cos 3\theta$$
$$+ [2a_4 + n_4 + a_3(a_1 + n_1) + a_1 n_3]\cos 4\theta$$
$$+ [2a_5 + n_5 + a_3(a_2 + n_2) + a_2 n_3]\cos 5\theta$$
$$+ [2a_6 + n_6 + \tfrac{1}{2}a_3^2 + a_3 n_3]\cos 6\theta$$
$$+ [2a_7 + n_7 + a_3(a_4 + n_4) + a_4 n_3]\cos 7\theta$$
$$+ [2a_8 + n_8]\cos 8\theta$$
$$= \frac{k\sqrt{(1-\nu)}}{a^{\frac{3}{2}}n} + m'\frac{k^2}{a^2 n}\int Srdt. \tag{9}$$

6. Remembering that $C = k\sqrt{[a(1-\nu)]}$, equation (4) also gives

$$r\left(\frac{dw}{dt}\right)^2 = \frac{1}{r^3}\left(r^2\frac{dw}{dt}\right)^2$$

$$= \frac{k^2}{r^3}\left(m'k\int Srdt + \sqrt{[a(1-\nu)]}\right)^2$$

$$= \frac{k^2}{r^3}a(1-\nu) + m'\frac{k^2}{r^3}\left[k\sqrt{[a(1-\nu)]}\,.\,2\int Srdt + m'k^2\left(\int Srdt\right)^2\right];$$

whence (5) becomes $\dfrac{d^2r}{dt^2} + \dfrac{k^2}{r^3}[r - a(1-\nu)] = m'k^2P,$

where $$P = R + \frac{1}{r^3}\left[k\sqrt{[a(1-\nu)]}\,.\,2\int Srdt + m'k^2\left(\int Srdt\right)^2\right]; \qquad (10)$$

or since $r = a(1+\sigma)$, $$\frac{d^2\sigma}{dt^2} + \frac{k^2}{a^3}\frac{\sigma+\nu}{(1+\sigma)^3} = m'\frac{k^2}{a}P. \qquad (11)$$

7. Differentiating twice with regard to t, since $\theta = (n'-n)t$, where n' is the mean motion of Titan, this gives

$$\frac{d^2\sigma}{dt^2} = -(n'-n)^2(a_1\cos\theta + 4a_2\cos 2\theta + 9a_3\cos 3\theta + \ldots). \qquad (12)$$

Expanding by the binomial theorem,

$$\frac{\sigma+\nu}{(1+\sigma)^3} = \nu + (1-3\nu)\sigma - 3(1-2\nu)\sigma^2 + 2(3-5\nu)\sigma^3 - 5(2-3\nu)\sigma^4$$
$$+ 3(5-7\nu)\sigma^5 - 7(3-4\nu)\sigma^6 + \ldots.$$

In the development of the third and higher powers of σ, it will be sufficient to put $\sigma = a_3\cos 3\theta$; whence

$$\left.\begin{aligned}
\sigma^3 &= \tfrac{3}{4}a_3^3\cos 3\theta + \tfrac{1}{4}a_3^3\cos 9\theta,\\
\sigma^4 &= \tfrac{3}{8}a_3^4 + \tfrac{1}{2}a_3^4\cos 6\theta + \tfrac{1}{8}a_3^4\cos 12\theta,\\
\sigma^5 &= \tfrac{5}{8}a_3^5\cos 3\theta + \tfrac{5}{16}a_3^5\cos 9\theta + \tfrac{1}{16}a_3^5\cos 15\theta,\\
\sigma^6 &= \tfrac{5}{16}a_3^6 + \tfrac{15}{32}a_3^6\cos 6\theta + \tfrac{3}{16}a_3^6\cos 12\theta + \tfrac{1}{32}a_3^6\cos 18\theta.
\end{aligned}\right\} \qquad (13)$$

8. If we neglect terms whose arguments are multiples of θ greater than 8θ, we have to terms of the fourth order

$$\frac{\sigma+\nu}{(1+\sigma)^3} = \nu - \tfrac{3}{2}a_3^2(1-2\nu) - \tfrac{15}{8}a_3^4(2-3\nu)$$

$$
\begin{aligned}
&+ [a_1(1-3\nu) - 3a_3(a_2+a_4)(1-2\nu)]\cos\theta \\
&+ [a_2(1-3\nu) - 3a_1a_3(1-2\nu)]\cos 2\theta \\
&+ [a_3(1-3\nu) + \tfrac{3}{2}a_3^3(3-5\nu)]\cos 3\theta \\
&+ [a_4(1-3\nu) - 3a_1a_3(1-2\nu)]\cos 4\theta \qquad (14)\\
&+ [a_5(1-3\nu) - 3a_2a_3(1-2\nu)]\cos 5\theta \\
&+ [a_6(1-3\nu) - \tfrac{3}{2}a_3^2(1-2\nu) - \tfrac{5}{2}a_3^4(2-3\nu)]\cos 6\theta \\
&+ [a_7(1-3\nu) - 3a_3a_4(1-2\nu)]\cos 7\theta \\
&+ \ a_8(1-3\nu)\cos 8\theta.
\end{aligned}
$$

Since the left hand members of (9) and (11) contain only terms involving cosines of multiples of θ and constant coefficients it is evident that the right hand members of the same contain similar terms only, and we may put

$$\frac{k^2}{a^2n(n'-n)}\int Srd\theta = \sum_1^\infty S_i\cos i\theta,$$

$$a^2P = \sum_0^\infty P_i\cos i\theta;$$

whence equations (9) and (11) may be written in the form

$$\frac{r^2}{a^2n}\frac{dw}{dt} = \frac{k\sqrt{(1-\nu)}}{a^{\frac{3}{2}}n} + m'\sum_1^\infty S_i\cos i\,\theta, \qquad (15)$$

$$\frac{a^3}{k^2}\frac{d^2\sigma}{dt^2} + \frac{\sigma+\nu}{(1+\sigma)^3} = m'\sum_0^\infty P_i\cos i\theta. \qquad (16)$$

9. With assumed values of our constants, R and S are computed for particular values of θ, selected at equal intervals from $\theta = 0$ to $\theta = 180°$, and then expanded mechanically into a series of cosines of multiples of θ, effected with constant coefficients; whence P and $\int Srd\theta$, and thence P_i and S_i are easily obtained.

If now we replace the left hand members of equations (15) and (16) by the developed expressions of (9) and (11), and equate the coefficients of the cosines of equal multiples of θ, there will result any number of equations from which may be derived a corresponding number of the quantities a_1, n_1, etc. Instead, however, of considering the mass of Titan as known, it will be better to assume a_3 as given, and to consider m' as one of the unknowns.

With Tisserand's values of the coefficients, the constant part of (16) gives an approximate value of ν; with this value of ν the constant part of (15) gives an approximate value of $\mu = a^3(n'-n)^2/k^2$; with these values of μ and ν, the terms of (16) involving 3θ give an approximate value of m'. On account of the impor-

tance of these quantities I have developed the equations giving μ, m', and ν to terms of the sixth, seventh, and eighth orders respectively. The remaining equations have been carried only to terms of the fourth order.

10. The equations giving ν, μ, and m' are as follows, including terms of the orders named:—

$$\nu\left[1 + 3(a_1^2 + a_2^2 + a_3^2 + a_4^2) + \tfrac{45}{8}a_3^4 + \tfrac{35}{4}a_3^6 + \tfrac{1575}{128}a_3^8\right] \tag{17}$$
$$= \tfrac{3}{2}(a_1^2 + a_2^2 + a_3^2 + a_4^2) + \tfrac{15}{4}a_3^4 + \tfrac{105}{16}a_3^6 + \tfrac{315}{32}a_3^8 + m'P_0,$$

$$\mu\left\{\begin{matrix} 1 + \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2 + a_4^2) \\ + a_1n_1 + a_2n_2 + a_3n_3 + a_4n_4 + \tfrac{1}{4}a_3^2n_6 \end{matrix}\right\}^2 = \left(\frac{n'-n}{n}\right)^2(1-\nu), \tag{18}$$

$$\begin{aligned} m'P_3 = {} & (1 - 3\nu - 9\mu)a_3 - 3(1-2\nu)a_1(a_2 + a_4) \\ & + \tfrac{9}{2}(1 - \tfrac{5}{3}\nu)a_3^3 + \tfrac{75}{8}(1 - \tfrac{7}{5}\nu)a_3^5 + \tfrac{245}{16}(1 - \tfrac{9}{7}\nu)a_3^7. \end{aligned} \tag{19}$$

Comparing (16) with (12) and (14) the coefficients of $\cos\theta$, $\cos 2\theta$, $\cos 4\theta$ give the following equations for the determination of a_1, a_2, a_3:—

$$\begin{aligned} (1 - 3\nu - \mu)a_1 - 3a_3(1-2\nu)a_2 - 3a_3(1-2\nu)a_4 &= m'P_1, \\ -3a_3(1-2\nu)a_1 + (1 - 3\nu - 4\mu)a_2 &= m'P_2, \\ -3a_3(1-2\nu)a_1 + (1 - 3\nu - 16\mu)a_4 &= m'P_4. \end{aligned} \tag{20}$$

Similarly, for $a_5, \ldots a_8$, we have

$$\begin{aligned} (1 - 3\nu - 25\mu)a_5 &= m'P_5 + 3a_3(1-2\nu)a_2, \\ (1 - 3\nu - 36\mu)a_6 &= m'P_6 + 3a_3(1-2\nu)\,.\,\tfrac{1}{2}a_3 + 5a_3^4(1 - \tfrac{3}{2}\nu), \\ (1 - 3\nu - 49\mu)a_7 &= m'P_7 + 3a_3(1-2\nu)a_4, \\ (1 - 3\nu - 64\mu)a_8 &= m'P_8. \end{aligned} \tag{21}$$

In the same way n_3 is obtained from

$$(1 + \tfrac{3}{4}a_3^2)n_3 = m'S_3 - 2a_3, \tag{22}$$

which is derived by comparing (15) with (9); after which n_1, n_2, and n_4 are obtained from

$$\begin{aligned} n_1 + a_3n_2 + a_3n_4 &= m'S_1 - 2a_1 - (a_3 + n_3)(a_2 + a_4), \\ a_3n_1 + n_2 &= m'S_2 - 2a_2 - (a_3 + n_3)a_1, \\ a_3n_1 + n_4 &= m'S_4 - 2a_4 - (a_3 + n_3)a_1; \end{aligned} \tag{23}$$

and the remaining terms give

$$\begin{aligned} n_5 &= m'S_5 - 2a_5 - (a_3 + n_3)\, a_2 - a_3 n_2, \\ n_6 &= m'S_6 - 2a_6 - \tfrac{1}{2}a_3^2 - a_3 n_3, \\ n_7 &= m'S_7 - 2a_7 - a_3\,(a_4 + n_4), \\ n_8 &= m'S_8 - 2a_8. \end{aligned} \qquad (24)$$

With the values of the constants thus obtained, new values are determined. This process must be repeated until the values obtained agree with those assumed.

11. For the determination of R and S were employed the following:—

$$n = 16^\circ\!.9199, \qquad n' = 22^\circ\!.5770,$$

$$\frac{r}{a} = 1 - 0.0004 \cos\ \theta - 0.0014 \cos 2\theta$$

$$+\ 0.1000 \cos 3\theta + 0.0006 \cos 4\theta,$$

$$w' - w = \theta - \ 10' \sin\ \theta - 13' \sin 2\theta$$

$$+\ 683' \sin 3\theta + \ 3' \sin 4\theta,$$

$$\log a = 0.0835122, \qquad r' = 1,$$

$$\rho^2 = r'^2 \sin^2(w' - w) + [r' \cos(w' - w) - r]^2,$$

$$h = \frac{1}{\rho^3} - \frac{1}{r'^3},$$

$$R = hr' \cos(w' - w) - \frac{r}{\rho^3},$$

$$S = hr' \sin(w' - w).$$

With these data values of a^2R and Sr were computed for every 20° from $\theta = 0$ to $\theta = 180°$, as follows:

θ	a^2R	Sr
0°	− 14.815	0.
20	− 3.547	+ 2.170
40	− 1.926	+ 0.340
60	− 1.501	− 0.134
80	− 1.130	− 0.623
100	− 0.438	− 1.182
120	+ 0.414	− 1.231
140	+ 0.975	− 0.699
160	+ 1.126	− 0.220
180	+ 1.133	0.

Whence

$$a^2R = -1.423 - 3.480\cos\theta - 1.538\cos 2\theta - 1.466\cos 3\theta$$
$$- 1.551\cos 4\theta - 1.322\cos 5\theta - 1.213\cos 6\theta$$
$$- 1.149\cos 7\theta - 1.116\cos 8\theta - 0.553\cos 9\theta,$$

$$Sr = -0.561\sin\theta + 0.800\sin 2\theta + 0.654\sin 3\theta$$
$$+ 0.311\sin 4\theta + 0.459\sin 5\theta + 0.368\sin 6\theta$$
$$+ 0.232\sin 7\theta + 0.131\sin 8\theta,$$

$$\int Srd\theta = + 0.561\cos\theta - 0.400\cos 2\theta - 0.218\cos 3\theta$$
$$- 0.078\cos 4\theta - 0.092\cos 5\theta - 0.061\cos 6\theta$$
$$- 0.033\cos 7\theta - 0.016\cos 8\theta.$$

Substituting in $a^3r^{-3} = 1 - 3\sigma + 6\sigma^2 - 10\sigma^3 + \ldots$

the values of σ, σ^2, etc., derived from (7), (8), and (13), employing Tisserand's values of a_1. a_2, a_3, etc., we have

$$a^3r^{-3} = 1.031 + 0.001\cos\theta + 0.004\cos 2\theta - 0.308\cos 3\theta$$
$$- 0.002\cos 4\theta - 0.001\cos 5\theta + 0.031\cos 6\theta$$
$$- 0.003\cos 9\theta.$$

Whence (10) gives

$$a^2P = -1.241 + 0.353\cos\theta - 4.399\cos 2\theta - 2.751\cos 3\theta$$
$$- 2.545\cos 4\theta - 1.457\cos 5\theta - 1.386\cos 6\theta$$
$$- 1.224\cos 7\theta - 1.172\cos 8\theta - 0.518\cos 9\theta.$$

Differentiating (1) and comparing with (3), we have

$$n_1 = + 0.0010,$$
$$n_2 = + 0.0025,$$
$$n_3 = - 0.1993,$$
$$n_4 = - 0.0012.$$

From equations (17) to (24) were next obtained the values given in the first column of the following table. With these and the values of P_i and S_i already

employed new values were obtained, which are given in the second column of the table:

	I	II
ν	0.014931	0.014091
log μ	9.054927	9.055352
$1/m'$	1290.	1370
a_1	− 0.0013	− 0.0012
a_2	− 0.0076	− 0.0071
a_3	+ 0.1000	+ 0.1000
a_4	+ 0.0027	+ 0.0025
a_5	+ 0.0018	+ 0.0017
a_6	+ 0.0003	+ 0.0003
a_7	0.0	0.0
a_8	+ 0.0002	+ 0.0002
n_1	+ 0.0027	+ 0.0025
n_2	+ 0.0137	+ 0.0129
n_3	− 0.1990	− 0.1990
n_4	− 0.0062	− 0.0058
n_5	− 0.0023	− 0.0022
n_6	− 0.0008	− 0.0008
n_7	+ 0.0002	+ 0.0002
n_8	− 0.0003	− 0.0003

The corresponding formulæ for the radius vector and longitude of Hyperion are

$$\begin{aligned}\frac{r}{a} = 1 &- 0.0012 \cos \theta - 0.0071 \cos 2\theta + 0.1000 \cos 3\theta \\ &+ 0.0025 \cos 4\theta + 0.0017 \cos 5\theta + 0.0003 \cos 6\theta \\ &+ 0.0002 \cos 8\theta,\end{aligned} \tag{25}$$

$$\begin{aligned} w = l &+ 26' \sin \theta + 66' \sin 2\theta - 682' \sin 3\theta \\ &- 15' \sin 4\theta - 5' \sin 5\theta - 1' \sin 6\theta.\end{aligned} \tag{26}$$

Finally from the formulæ

$$k = \frac{n'}{\sqrt{(1 + m')}}, \quad a^3 = \frac{\mu k^2}{(n' - n)^2}, \tag{27}$$

and the values of μ and m' obtained from the second column of the above table, we have

$$\log k = 1.353507, \quad \log a = 0.085727.$$

12. As a check, the computations were carried through a second time, this time dividing the circle into 72 parts, and working independently of Tisserand's values of the smaller coefficients in the expressions for r/a and dw/dt. In computing the forces, I have assumed

$$r = a\,(1 + 0.1 \cos 3\theta),$$

$$w' - w = \theta + 683' \sin 3\theta.$$

To find a, if we neglect small terms of the sixth order and those containing m', we have

$$n_3 = -\frac{2a_3}{1 + \frac{3}{4}a_3^2}, \tag{28}$$

$$\nu = \frac{\frac{3}{2}a_3^2}{1 + \frac{1}{2}a_3^2}, \tag{29}$$

$$\alpha^3 = \frac{a'^3}{a^3} = \frac{n^2}{n'^2} \cdot \frac{(1 + \frac{1}{2}a_3^2 + a_3 n_3)^2}{1 - \nu}; \tag{30}$$

whence

$$n_3 = 0.1485, \qquad \nu = 0.01493,$$

$$a' = 1.00000, \quad \log a = 0.08569.$$

Including only those terms which will be needed in computing m', we have

$$a^2R = -\ 1.280 - 1.141 \cos 3\theta - \ldots,$$

$$\int Srd\theta = -\ 0.206 \cos 3\theta - 0.083 \cos 6\theta - 0.038 \cos 9\theta - \ldots,$$

$$a^3r^{-3} = \quad 1.031 - 0.308 \cos 3\theta + 0.031 \cos 6\theta - \ldots;$$

$$a^2P = -\ 1.061 - 2.600 \cos 3\theta - \ldots.$$

Finally, the equation

$$[3a_3P_0 + (1 + \tfrac{1}{2}a_3^2)P_3]\,m' = \left[\frac{1}{\sqrt{(1 - a_3^2)}} - 9\mu(1 + \tfrac{1}{2}a_3^2)\right]a_3, \tag{31}$$

obtained by substituting (17) in (19) and neglecting small terms of the sixth order, gave

$$m' = \tfrac{1}{1379}.$$

13. This approximate value for the mass of Titan is about eight times that given by Tisserand. The reasons for this are apparent: To terms of the first order, $\mu = (n' - n)^2 / n^2$ and $m'P_3 = (1 - 9\mu)\, a_3$, P_3 being obtained from the hypothesis $r = a$ and $r' = a'$. $m'P_3$, however, is so small that the introduction of a more accurate value of μ and the addition of the remaining terms given in equation (19) change it ($m'P_3$) from -0.0006 to -0.0021, while P_3 changes from -6.54 to -2.60. The disturbance produced by Titan is, of course, greatest near conjunction; but Hyperion is then always at apo-saturnium, and accordingly the average distance between the two satellites is approximately $a\,(1 + a_3) - a'$ instead of $a - a'$; i. e. to terms of the first order, $0.275a$ instead of $0.175a$.

14. Prof. Newcomb in a discussion of the motion of the peri-saturnium of Hyperion*, assuming, as I have done, that the orbit of Titan is a circle, has found $m' = \frac{1}{12800}$; but in so doing has assumed n to be the *elliptic* mean motion, instead of the mean motion in *longitude*: whereas the mean value of the mean motion in the variable ellipse is, approximately, $k / a^{\frac{3}{2}}$. Assuming

$$k = n' \quad \text{and} \quad \mathrm{n} = \frac{k}{a^{\frac{3}{2}}} = 16^\circ.7931,$$

we have

$$4\mathrm{n} - 3n' = -0^\circ.5585\,;$$

whence, using the same values for the forces as those employed by Newcomb,

$$m' = \frac{4\mathrm{n} - 3n'}{40.6\,\mathrm{n}} = \frac{1}{1209},$$

which agrees approximately with the values already found.

*On the Motion of Hyperion. A New Case in Celestial Mechanics, pp. 365 7.

[TO BE CONTINUED.]